Note on central limit theorem

The *central limit theorem* states that under conditions weak enough to cover random variables $\mathbf{x}$ with almost any distribution likely to occur in nature, the random variable

$$\mathbf{y} = n^{-1/2} \sum_{k=1}^{n} (\mathbf{x}_k - m) \tag{1}$$

whose mean is zero and whose variance is σ^2, has the limit $n \to \infty$ a Gaussian probability density function.

Here is the proof of the central limit theorem. We form a characteristic function of $\mathbf{y}$ and show that as n goes to infinity, it approaches more and more closely the characteristic function of a Gaussian random variable:

$$\Phi_{\mathbf{y}}(\omega) = E(e^{j\omega \mathbf{y}}) = E \exp\left[j\omega n^{-1/2} \sum_{k=1}^{n} (\mathbf{x}_k - m) \right] \tag{2}$$

$$= \prod_{k=1}^{n} E \exp[j\omega n^{-1/2} (\mathbf{x}_k - m)] \tag{3}$$

The characteristic function $\Phi_{\mathbf{x}}(\omega)$ can be expanded in Taylor series

$$\Phi_{\mathbf{x}}(\omega) = \sum_{k=0}^{\infty} \frac{(j\omega)^k}{k!} E((\mathbf{x} - m)^k) \tag{4}$$

and therefore

$$\Phi_{\mathbf{y}}(\omega) = \left(\sum_{l=0}^{\infty} \frac{(j\omega n^{-1/2})^l}{l!} E((\mathbf{x} - m)^l) \right)^n \tag{5}$$

Then

$$\lim_{n \to \infty} \Phi_{\mathbf{y}}(\omega) = \lim_{n \to \infty} \left(1 + \frac{x}{n} \right)^n = e^x \tag{6}$$

where

$$x = -\frac{1}{2} \sigma^2 \omega^2 \tag{7}$$

and

$$\Phi_{\mathbf{y}}(\omega) = e^{-\frac{1}{2}\sigma^2\omega^2} \tag{8}$$

which is the characteristic function of Gaussian random variable with mean zero and variance σ^2.

References

[1] Carl W. Helstrom, *Probability and Stochastic Processes for Engineers* Macmillan Publishing Company, New York, Collier Macmillan Publishers, London, 1984.

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