Note on Cauchy-Schwarz inequality

x, y are vectors in n-dimensional space of the set of real numbers $\mathbb{R}$ what we write as $x, y \in \mathbb{R}^n$. In other words

$$x = \sum_{i=1}^{n} x_i \mathbf{e}_i \tag{1}$$
$$y = \sum_{j=1}^{n} y_j \mathbf{e}_j$$

where $\mathbf{e}_i$ is the i-th basis vector, and x_i is the i-th component of the vector x. If basis vectors are orthonormal then we have

$$x \cdot y = \left(\sum_{i=1}^{n} x_i \mathbf{e}_i \right) \cdot \left(\sum_{j=1}^{n} y_j \mathbf{e}_j \right) = \sum_{i=1}^{n} x_i y_i \tag{2}$$

because the products

$$\mathbf{e}_i \cdot \mathbf{e}_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{3}$$

If x and y are linearly independent vectors, then we have for all $\lambda \in \mathbb{R}$

$$\begin{aligned} 0 < |\lambda y - x|^2 = \sum_{i=1}^{n} (\lambda y_i - x_i)^2 &= \sum_{i=1}^{n} \left(\lambda^2 y_i^2 - 2\lambda x_i y_i + x_i^2 \right) \\ &= \lambda^2 \sum_{i=1}^{n} y_i^2 - 2\lambda \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} x_i^2 \end{aligned} \tag{4}$$

The last equation in (4) is a quadratic equation for unknown λ. If a quadratic expression $a\lambda^2 + b\lambda + c$ has to be greater than zero, then its delta has to be less than zero as stated in [1] on page 2

$$\Delta = b^2 - 4ac < 0 \tag{5}$$

what gives

$$\Delta = 4 \left(\sum_{i=1}^{n} x_i y_i \right)^2 - 4 \left(\sum_{i=1}^{n} x_i^2 \right) \left(\sum_{i=1}^{n} y_i^2 \right) < 0 \tag{6}$$

and therefrom

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 < \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \tag{7}$$

what gives

$$(x \cdot y)^2 < x^2 y^2 \tag{8}$$

which after taking square root of both sides is equivalent to

$$|x \cdot y| < |x||y| \tag{9}$$

If we take into account the situation when the vectors x, y are linearly dependent, then we have to consider that $\Delta = 0$. We arrive at the general situation for all vectors $x, y \in \mathbb{R}^n$ that

$$|x \cdot y| \leq |x||y| \tag{10}$$

The inequality (10) is called Cauchy-Schwarz inequality.

References

[1] Spivak, M. (1998) *Calculus on Manifolds* Westview Press

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