On Buffon's needle problem

The original statement of the Buffon's needle problem was: "Suppose we have a floor made of parallel strips of wood, each the same width, and we drop a needle onto the floor. What is the probability that the needle will lie across a line between two strips?"

We may state the problem also in other words. The needle of length l is being thrown on a paper sheet on which parallel lines are drawn and the distance between each two nearest parallel lines is $d > l$ (see Figure 1). What is the probability that the needle lies on a line?

The needle crosses the line when the needle starting point has the distance less than $l|\cos(\Theta)|$ from a line.

In general the needle starting point can lie anywhere between the lines or on the lines, i.e. on the distance of length d between the lines.

So the probability for a particular angle Θ that the needle crosses the line is

$$P(\Theta) = \frac{l|\cos(\Theta)|}{d} \tag{1}$$

We can always write then

$$P(\Theta) = \frac{l|\cos(\Theta)|}{d} = \frac{l|\cos(\Theta)|\,\Delta\Theta}{d\,\Delta\Theta} \tag{2}$$

what corresponds to the ratio of the area $l|\cos(\Theta)|\,\Delta\Theta$ for the angle Θ to the area $d\Delta\Theta$ (see Figure 2).

If we want to compute the probability that the needle will cross a line taking into account all possible angle values Θ we need to take into account all such possible ratios like the one in equation (2) with $\Delta\Theta \to 0$, i.e. we have to perform the appropriate integration obtaining in general the probability P of the situation that the needle will cross the line

$$P = \frac{\int_0^{2\pi} l|\cos(\Theta)|\,d\Theta}{\int_0^{2\pi} d\,d\Theta} = \frac{4l\int_0^{\pi/2}\cos(\Theta)\,d\Theta}{2\pi d} = \frac{4l}{2\pi d} = \frac{2l}{\pi d} \tag{3}$$

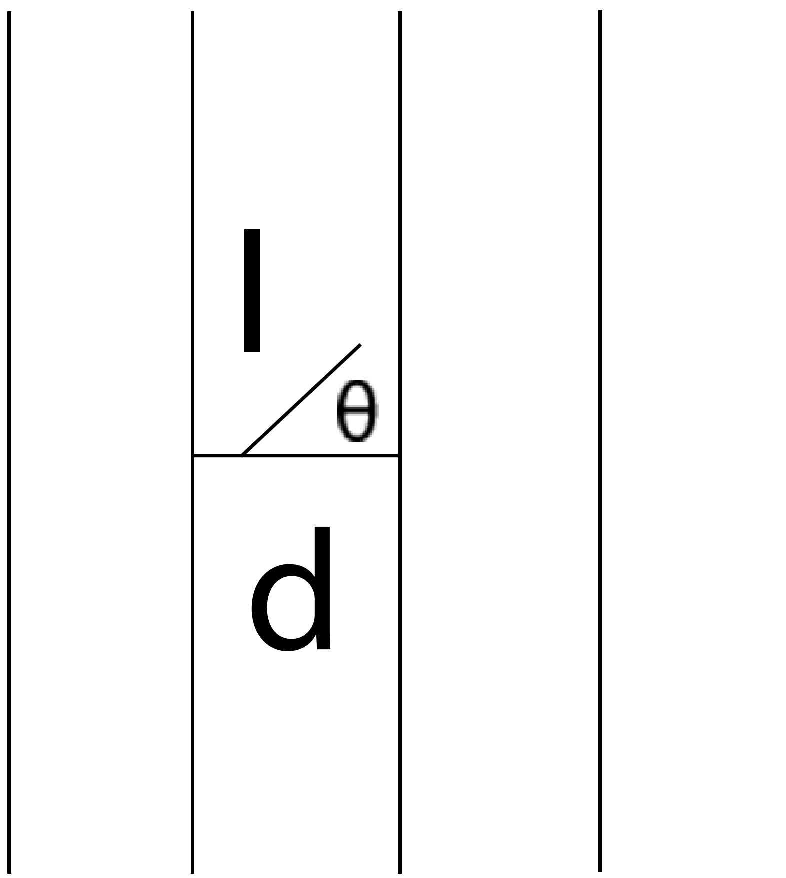

Figure 1: Needle of length l at angle Θ to the section of distance d between parallel lines.

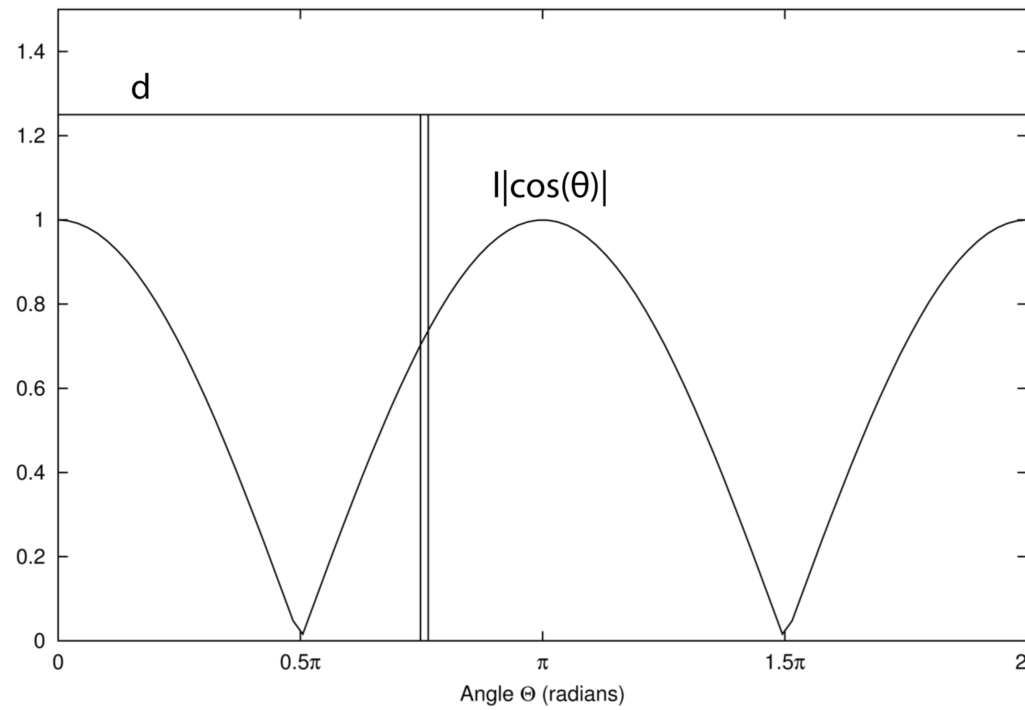

Figure 2: Plot of $l|\cos(\Theta)|$ with a narrow stripe of width $\Delta\Theta$ at a particular angle value Θ. The probability that the needle will cross a line when it falls at a particular angle value Θ is the ratio $\frac{l|\cos(\Theta)|}{d} = \frac{l|\cos(\Theta)|\,\Delta\Theta}{d\,\Delta\Theta}$ what corresponds to the ratio of the area $l|\cos(\Theta)|\,\Delta\Theta$ to the area $d\,\Delta\Theta$ for the angle Θ. To compute the probability that the needle crosses the line in general for any angle Θ we need to integrate and take the ratio of the area under the curve $l|\cos(\Theta)|$ which corresponds to all situations when the needle crosses a line at any angle Θ to the area $2\pi d$ which corresponds to any situation of a needle lying on a line or between lines.

References

[1] https://en.wikipedia.org/wiki/Buffon%27s_needle

[2] http://mathworld.wolfram.com/BuffonsNeedleProblem.html

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