The basic trigonometric system. Orthogonality of sines and cosines

By the basic trigonometric system we mean the system of functions

$$1, \cos x, \sin x, \cos 2x, \sin 2x, \ldots, \cos nx, \sin nx, \ldots \tag{1}$$

All these functions have the common period 2π (although cos nx and sin nx also have the smaller period $2\pi/n$).

For any integer $n \neq 0$

$$\int_{-\pi}^{\pi} \cos nx\, dx = \left[\frac{\sin nx}{n}\right]_{x=-\pi}^{x=\pi} = 0 \tag{2}$$

$$\int_{-\pi}^{\pi} \sin nx\, dx = \left[-\frac{\cos nx}{n}\right]_{x=-\pi}^{x=\pi} = 0$$

We use the relationships

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \tag{3}$$

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \tag{4}$$

to obtain

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)] \tag{5}$$

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$

and the relationships

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta \tag{6}$$

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta \tag{7}$$

to obtain

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)] \tag{8}$$

Using the relationships (5) and (8) we can compute for integers $m \neq n$

$$\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x + \cos(m+n)x]\, dx = 0 \tag{9}$$

$$\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\cos(m-n)x - \cos(m+n)x]\, dx = 0$$

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} [\sin(m-n)x + \sin(m+n)x]\, dx = 0$$

and

$$\int_{-\pi}^{\pi} \cos^2 nx\, dx = \frac{1}{2}\int_{-\pi}^{\pi} [1+\cos 2nx]\, dx = \pi \tag{10}$$

$$\int_{-\pi}^{\pi} \sin^2 nx \, dx = \frac{1}{2} \int_{-\pi}^{\pi} [1 - \cos 2nx] \, dx = \pi$$

The formulas (2), (9) and (10) show that the integral over the interval $[-\pi, \pi]$ of the product of any two *different* functions of the system (1) vanishes.

We say that two functions $\varphi(x)$ and $\psi(x)$ are orthogonal on interval $[a, b]$ if

$$\int_{a}^{b} \varphi(x)\psi(x) \, dx = 0 \tag{11}$$

We see that the functions of the system (1) are pairwise orthogonal on the interval $[-\pi, \pi]$ or more briefly, that the system (1) is *orthogonal* on $[-\pi, \pi]$.

References

[1] G.P. Tolstov, *Fourier Series* Translation by R.A. Silverman, Dover Publications, Inc., New York NY, 1976.

Pawel Jan Piskorz (paweljs@gmail.com)