On solution of diffusion equation with drift

Diffusion equation with drift in one dimension

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} \tag{1}$$

is an example of a Fokker-Planck equation. It may be not obvious to find the solution of the diffusion equation with drift just by looking at the equation itself. The solution may come to mind however if one knows the derivation of this equation. One can expect that the solution is like for the diffusion equation without drift

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} \tag{2}$$

however when drift at the velocity v is present, the concentration distribution $C(x,t)$ obtained from the simple diffusion equation (2) moves at the velocity v in the direction of the x axis towards increasing values of x. Knowing the solution $C(x,t)$

$$C(x,t) = \frac{C_0}{\sqrt{2\pi}\sqrt{2Dt}} e^{-x^2/4Dt} \tag{3}$$

of the simple diffusion equation (2) in one dimension with the initial condition

$$C(x,0) = C_0\,\delta(x) \tag{4}$$

where $\delta(x)$ is the Dirac's delta function we can propose the solution of the diffusion equation with drift (1) as

$$C(x,t) = \frac{C_0}{\sqrt{2\pi}\sqrt{2Dt}} e^{-(x-vt)^2/4Dt} \tag{5}$$

which is just the solution of simple diffusion equation (2) however this solution is shifted along the x axis by the distance vt to the right. Differentiating the solution (5) and solving

$$\frac{\partial}{\partial t}\left(\frac{C_0}{\sqrt{2\pi}\sqrt{2Dt}}e^{-(x-vt)^2/4Dt}\right) = 0 \tag{6}$$

we arrive at the quadratic equation

$$v^2t^2 + 2Dt - x^2 = 0 \tag{7}$$

The discriminant Δ of equation (7) is

$$\Delta = 4D^2 + 4v^2x^2 \tag{8}$$

and the root

$$t = \frac{\sqrt{D^2 + v^2x^2} - D}{v^2} \tag{9}$$

of the equation (7) gives time t of the maximum concentration $C(x,t)$ occurring at point x.

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