# On derivation of diffusion equation with drift

The flux $F$ of $N$ particles which flow to the right with velocity $v$

$$v = \frac{\Delta x}{\Delta t} \tag{1}$$

in one dimension from the point $x$ into the area from $x$ to $x + \Delta x$ in time $\Delta t$ through the cross section of area $a$ is

$$F = \frac{N(x,t)}{a\Delta t} = \frac{N(x,t)}{a\Delta x} v = C(x,t)v \tag{2}$$

where $C(x,t)$ is the concentration of the particles at point $x$ at time $t$ when $\Delta x \to 0$. The ratio $N(x,t)/a\Delta x$ is the ratio of $N(x,t)$ particles in the volume element $a\Delta x$ per the volume of that volume element, i.e. the average concentration of particles in the volume element of cross section of area $a$ and length $\Delta x$ extending from $x$ to $x + \Delta x$. If one takes a limit $\lim_{\Delta x \to 0} N(x,t)/a\Delta x$ one receives the concentration $C(x,t)$ of particles at point $x$ at time $t$

$$\lim_{\Delta x \to 0} \frac{N(x,t)}{a\Delta x} = C(x,t) \tag{3}$$

The total flux due to the concentartion gradient $\frac{\partial C}{\partial x}$ and the drift is

$$F = -D\frac{\partial C}{\partial x} + vC \tag{4}$$

The total flux in three dimensions equals

$$\vec{F} = -D\nabla C + \vec{v}C \tag{5}$$

The equation (5) describes Fick's law with drift. Taking into account the continuity equation in three dimensions

$$\frac{\partial C}{\partial t} + \nabla \cdot \vec{F} = 0 \tag{6}$$

we obtain

$$\frac{\partial C}{\partial t} = \nabla \cdot (D\nabla C - C\vec{v}) \tag{7}$$

We use the relation

$$\operatorname{div} C\vec{v} = \nabla \cdot C\vec{v} = C\nabla \cdot \vec{v} + \vec{v} \cdot \nabla C \tag{8}$$

If the diffusion coefficient does not depend on the space coordinates, and because the number of particles which flow into a volume element due to drift is equal to the number of particles which flow out of this volume element due to drift, we have $\nabla \cdot \vec{v} = 0$, and we receive

$$\frac{\partial C}{\partial t} = D\nabla^2 C - \vec{v} \cdot \nabla C \tag{9}$$

In one dimension we have the equation

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - v\frac{\partial C}{\partial x} \tag{10}$$

which is an example of Fokker-Planck equation. Equation (10) is known as diffusion equation with drift.

Pawel Jan Piskorz (paweljs@gmail.com)